Journal of Statistical Physics, Vol. 71, Nos. 3/4, 1993

On the Determination of Box Dimensions by Means of Wavelet Transforms

H. O. Rasmussen[1]

Sufficient conditions are established for the determination of box dimensions of graphs from the decrease of the corresponding wavelet transforms. As an application, wavelet Weierstrass functions are constructed.

KEY WORDS: Wavelet transforms; fractals; box dimensions.

Wavelet transforms decompose square-integrable functions in terms of translated and scaled versions of an "analyzing wavelet," a square-integrable function. The wavelet coefficients used for this decomposition are formed by convoluting the original function with translated and scaled versions of the analyzing wavelet. Since the analyzing wavelet has zero integral, the wavelet coefficients depend only on local averages of fluctuations in the function (provided the decay of the analyzing wavelet at infinity is sufficiently rapid). It is well known that fluctuations in functions are related to the box dimensions of the graphs.[2] But, is it also possible to determine these dimensions by means of the wavelet coefficients?

Such relations are by now well established for measures.[1,3–6,9] The problem for continuous functions is still unsolved, but a first step was taken in refs. 5 and 6, where the decrease of wavelet transforms was related to the Hölder continuity of functions. Specifying the Hölder continuity, however, does not determine the value of box dimensions—it merely provides an upper bound (see Proposition 2 below). In the present paper, lower bounds on box dimensions are obtained by exploiting the observation that a local lower bound on the wavelet transform leads to a lower bound on fluctuations in the function in an appropriately chosen interval.

[1] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, CB3 9EW Cambridge, England.

The lower bounds thus obtained are equal to the upper bounds, showing that the wavelet coefficients determine the box dimension uniquely.

Box dimensions are easy to compute and are therefore often used to characterize scale invariance, though, from a mathematical viewpoint, they do have certain unpleasant properties.[2] The following result is often useful for determining box dimensions:[2]

Proposition 1. Suppose that $\mathbb{F} \subseteq \mathbb{R}^2$ is intersected by n_k δ_k-mesh cubes with $\delta_k \searrow 0$ as $k \to \infty$, and $\delta_{k+1} \geqslant c\delta_k$ for $1 > c > 0$. Then the box dimensions of $\mathbb{F}$ is given by

$$D = \lim_{k \to \infty} \frac{\log n_k}{-\log \delta_k} \tag{1}$$

provided the limit exists.

A function f: $\mathbb{R} \to \mathbb{R}$ is Hölder continuous[6,8] with exponent α, where $0 < \alpha \leqslant 1$, if there exists a $C > 0$ such that

$$\sup_{|x-y| \leqslant h} |f(x) - f(y)| \leqslant Ch^{\alpha} \tag{2}$$

for all $h > 0$. For bounded functions, this condition holds for all $h > 0$ if and only if it holds for all $h > 0$ sufficiently small (though generally with a different constant). Rather than counting boxes directly, we rely on the following result,[2] relating Hölder exponents to box dimensions:

Proposition 2. Let f: $\mathbb{I} \to \mathbb{R}$ be a continuous function, where $\mathbb{I}$ is a bounded interval.

(a) If f is Hölder continuous with exponent α and the box dimension D exists, then $D \leqslant 2 - \alpha$.

(b) Suppose that there are numbers $K_2 > 0$ and $0 \leqslant \alpha \leqslant 1$, and a decreasing sequence as in Proposition 1, with the following property: for each $t_1 \in \mathbb{I}$ and $k \in \mathbb{N}$, there exists $t_2 \in \mathbb{I}$ such that $|t_1 - t_2| \leqslant 2\delta_k$ and

$$|f(t_1) - f(t_2)| \geqslant K_2 \delta_k^{\alpha} \tag{3}$$

If the box dimension exists, then $D \geqslant 2 - \alpha$.

Remark. The box dimension exists if the α in Eq. (3) may be chosen equal to the Hölder exponent. "The use of $2\delta_k$, rather than δ_k, is for later convenience."

Assume that the function to be analyzed is real-valued and bounded.

The wavelet transform is then formed by convoluting the function with wavelets of constant shape in the following way (see, for instance, ref. 6):

$$W(\lambda, r) = \frac{1}{\lambda} \int_{-\infty}^{+\infty} f(t)\, g^*\left(\frac{t-r}{\lambda}\right) dt \tag{4}$$

where r is a real number, λ is positive, and the complex conjugate of $g(t)$ is denoted by $g^*(t)$. The wavelet transform is invertible, but this property is not used in the present paper. Here, the analyzing wavelet must be continuously differentiable and furthermore satisfy the following conditions:

1. There exist $C > 0$, $C' > 0$, $\varepsilon > 0$, and $m > 0$ such that

$$|g(t)| \leqslant C(1+|t|)^{-m-2} \qquad \text{and} \qquad |g'(t)| \leqslant C'(1+|t|)^{-\varepsilon-2}$$

2. The following relations hold:

$$\int_{-\infty}^{+\infty} g(t)\, dt = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} tg(t)\, dt = 0$$

The analyzing wavelet may be complex-valued.

The problem of determining local Hölder exponents by means of wavelet transforms was first considered by Holschneider,[5] and the results proved in that paper were later extended by Holschneider and Tchamitchian.[6] The following result was stated under more restrictive conditions in refs. 5 and 6, but the following version is more suitable for our purposes and is readily proved by the methods outlined there (this theorem, by the way, is a wavelet adaption of a classical result on the characterization of Hölder continuity).

Theorem 3. Let the analyzing wavelet satisfy the above conditions with $m > \alpha$, where $\alpha \in]0, 1[$. Then the function is Hölder continuous with exponent α if and only if

$$W(\lambda, r) = O(\lambda^\alpha) \tag{5}$$

holds uniformly in r.

Remark. If Eq. (5) holds uniformly, then the boundedness of the function (assumed above) implies that

$$|W(\lambda, r)| \leqslant C\lambda^\alpha \tag{6}$$

for some $C > 0$ and all $r \in \mathbb{R}$.

It is often difficult to find lower bounds on $|f(t_1)-f(t_2)|$ for given $|t_1-t_2|$ because these lower bounds generally do not hold at every point. A similar problem with lower bounds is encountered in the asymptotic behavior of entire functions. In this case the problem is to obtain a lower bound on the maximum value of the function on a circle of given radius in the complex plane and centered at the origin. This is possible if the integral of the function along the circle can be estimated, because then the mean value theorem for integrals can be used: if $\int_a^b f(t)\,dt > c$, where $b > a$, then there is at least one point $t_0 \in \,]a, b[$ such that $f(t_0) > c/(b-a)$. This trick[2] is used to prove the following theorem.

Theorem 4. Suppose that $f\colon [a, b] \to \mathbb{R}$ is square-integrable and that the analyzing wavelet g satisfies the above conditions. Let $\{\lambda_k\}$, $k = 0, 1, \ldots$, be a decreasing sequence with $\lambda_k \searrow 0$ as $k \to \infty$, and $\lambda_{k+1} \geqslant C\lambda_k$ for some $1 > C > 0$ and all $k \in \mathbb{R}$. Suppose that there is a collection of points $\{r_{kl}\}$, a D such that $\max\{1, 2-m\} < D < 2$, and $\alpha, \beta > 0$ with the following properties:

(a) For each $t_0 \in [a, b]$ and for some choice of k and l,

$$|t_0 - r_{kl}| < \alpha\lambda_k \tag{7}$$

(b) For all k and l,

$$|W(\lambda_k, r_{kl})| \geqslant \beta\lambda_k^{2-D} \tag{8}$$

(c) And

$$|W(\lambda, r)| = O(\lambda^{2-D}) \tag{9}$$

uniformly in r.

Then the box dimension of the graph exists and is equal to D.

Remarks. The sequence $\{r_{kl}\}$ may be finite for each fixed k.

The following proof does not require that r and λ are continuous parameters. Theorem 4, therefore, also holds for discrete wavelet transforms. Notice that α no longer denotes the Hölder exponent.

Proof. To use the mean value theorem for integrals, notice first that

$$\begin{aligned}\int_{-\delta}^{+\delta} |f(t+t_0)-f(t_0)|\,dt \\ \geqslant \frac{\lambda}{\sup|g|}\frac{1}{\lambda}\int_{-\delta}^{+\delta} |f(t+t_0)-f(t_0)| \cdot \left| g\left(\frac{t}{\lambda}\right)\right| dt\end{aligned} \tag{10}$$

[2] Littlewood was apparently the first to realize the usefulness of this trick; the example mentioned here is part of his first paper.[7]

The norm-inequality for integrals gives a lower bound on the right-hand side of this inequality,

$$\frac{\lambda}{\sup|g|} \frac{1}{\lambda} \int_{-\delta}^{+\delta} |f(t+t_0)-f(t_0)| \cdot \left| g\left(\frac{t}{\lambda}\right) \right| dt$$

$$\geqslant \frac{\lambda}{\sup|g|} \left| \frac{1}{\lambda} \int_{-\delta}^{+\delta} [f(t+t_0)-f(t_0)]\, g^*\left(\frac{t}{\lambda}\right) dt \right| \tag{11}$$

$$= \frac{\lambda}{\sup|g|} \left| W(\lambda, t_0) - \frac{1}{\lambda} \int_{\mathbb{R}-[-\delta,+\delta]} [f(t+t_0)-f(t_0)]\, g^*\left(\frac{t}{\lambda}\right) dt \right| \tag{12}$$

The last step followed by using condition 2, which says that the analyzing wavelet has zero integral. Suppose first that $t_0 = r_{kl}$ for some choice of k and l. The last expression in (12) can then be evaluated by using the Hölder continuity of $f(t)$ and the decay at infinity of the analyzing wavelet. A bound for the last integral in Eq. (12) can be found by using Theorem 3 and Eq. (9), which together imply that $2-D$ is a Hölder exponent for the function $f(t)$, and condition 1, which determines the decay at infinity for the analyzing wavelet. Notice first that

$$\frac{1}{\lambda} \int_{\delta}^{\infty} |t|^{2-D} \left| g^*\left(\frac{t}{\lambda}\right) \right| dt \leqslant \frac{K}{m-2+D} \lambda^{m+1} \delta^{1-D-m} \tag{13}$$

for some constant $K > 0$. To use this upper bound on the last integral in Eq. (12), the wavelet transform $|W(\lambda, t_0)|$ in Eq. 12 must be sufficiently large; this happens when $t_0 = r_{kl}$. It therefore follows that

$$\int_{-\delta}^{+\delta} |f(t+r_{kl})-f(r_{kl})|\, dt \geqslant \frac{\lambda}{\sup|g|} \left|\, |W(\lambda, r_{kl})| - C\lambda^{1+m}\delta^{1-D-m} \right| \tag{14}$$

for some constant $C > 0$. To evaluate the right-hand side in Eq. (14), let $\delta = \delta_k = \lambda_k \alpha h$. The lower bound on the wavelet transform in Eq. (8) then implies that

$$\left|\, |W(\lambda_k, r_{kl})| - C\lambda_k^{1+m}\delta_k^{1-D-m} \right| \geqslant |\beta\alpha^{D-2}h^{D-2} - C\lambda^{-m}h^{-m}|\, \delta_k^{2-D} \tag{15}$$

provided h is sufficiently large; notice that h may be chosen independently of k and l. By combining Eqs. (14) and (15), and then using the mean value theorem for integrals, it follows that there is a $t_1 \in\,]r_{kl} - \delta_k; r_{kl} + \delta_k[$ such that

$$|f(t_1 + r_{kl}) - f(r_{kl})| > \frac{1}{1\alpha h \sup|g|} |\beta\alpha^{D-2}h^{D-2} - C\alpha^{-1-m}h^{-1-m}|\, \delta_k^{2-D} \tag{16}$$

For given $t_0 \in [a, b]$ and $k \in \mathbb{N}$, it follows from Eq. (7) that there is an $r_{kl} \in [a, b]$ such that $|t_0 - r_{kl}| < \delta_k/h$, implying that $|t_0 - t_1| < 2\delta_k$, provided $h \geqslant 1$. Choose now a $t_1 \in]r_{kl} - \delta_k; r_{kl} + \delta_k[$ such that Eq. (16) holds. For either $t_2 = r_{kl}$ or $t_2 = t_1$, it follows that

$$|f(t_0) - f(t_2)| \geqslant \frac{1}{2\alpha h \sup |g|} |\beta\alpha^{D-2}h^{D-2} - C\alpha^{-1-m}h^{-1-m}| \, \delta_k^{2-D} \tag{17}$$

and $|t_0 - t_2| < 2\delta_k$. This implies that condition (b) of Proposition 2 is satisfied and therefore that the box dimension is larger than or equal to D. Condition (a) of Proposition 2 is satisfied because the function is Hölder continuous with exponent $2 - D$, as remarked above, and the box dimension therefore exists and is equal to D. ∎

The above theorem leads to a generalization of the classical Weierstrass functions (see, for instance, ref. 2). Weierstrass[10] introduced these functions as examples of continuous functions that are nowhere differentiable, but nowadays they are mainly used as examples of functions that have "fractal" graphs.[2] To generalize such functions to bases that consist of wavelets rather than sine waves, let

$$\psi_{ij}(t) = 2^{j/2}\psi(2^j t - i)$$

where $i, j \in \mathbb{Z}$, be an orthonormal basis for $L^2(\mathbb{R})$ (for the construction of such bases, see ref. 8). Then any square-integrable function can be written in the form

$$f(t) = \sum_{ij} a_{ij}\psi_{ij}(t)$$

where the coefficients a_{ij} are the inner products between $f(t)$ and the functions in the basis. Suppose now that $\psi(t)$ has compact support and that the a_{ij} vanish whenever the product $2^{-j}i$ is sufficiently large. The function $f(t)$ then has compact support; say in $[a, b]$. Let $\{n_i\}_{i \in \mathbb{N}}$ be an unbounded sequence of integers such that

$$n_{i+1} > n_i$$

$$|n_i - n_{i+1}| \leqslant \alpha$$

for some $\alpha > 0$ and all $i \in \mathbb{N}$. For each $t_0 \in [a, b]$, it is then possible to choose i and j such that

$$|t_0 - 2^{-j}n_i| \leqslant |2^{-j}n_{i+1} - 2^{-j}n_i| \leqslant 2^{-j}\alpha$$

This shows that condition (a) in Theorem 4 is satisfied. If there also exist positive constants M_1 and M_2 such that

$$M_1 \leqslant 2^{(2-D)j + j/2}|a_{n_i j}| \leqslant M_2 \tag{18}$$

then the graph of

$$f(t) = \sum_{n_i j} \psi_{n_i j}(t) \tag{19}$$

has box dimensional equal to D. This function may be described as a *lacunary* wavelet series. Results in ref. 8 then show that the function in Eq. (19) is continuous and nondifferentiable at every point. Notice that the nondifferentiability and the dimension remain unchanged when $f(t)$ is extended to a function of the form

$$f(t) = \sum_{kl} a_{kl} \psi_{kl}(t)$$

where a_{ij} satisfies the inequalities in (18) whenever $k = n_i$ for some $i \in \mathbb{N}$, and at least the second inequality for all other $k \in \mathbb{N}$.

ACKNOWLEDGMENTS

The author thanks Carlsbergfondet, Schlumberger Research Cambridge, and the Danish Research Academy for financial support.

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